

On the accuracy of conservation of adiabatic invariants in slow-fast systems

Tan Su*

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Abstract

Let the adiabatic invariant of action variable in slow-fast Hamiltonian system with two degrees of freedom have two limiting values along the trajectories as time tends to infinity. The difference of two limits is exponentially small in analytic systems. An iso-energetic reduction and canonical transformations are applied to transform the slow-fast systems to form of systems depending on slowly varying parameters in a complexified phase space. On the basis of this method an estimate for the accuracy of conservation of adiabatic invariant is given for such systems.

1 Introduction

Consider a Hamiltonian system with two degrees of freedom. The Hamiltonian E depends slowly on coordinate $\varepsilon^{-1}x$ and fast on coordinate q :

$$E = E(p, q, y, x)$$

where $q, \varepsilon^{-1}x$ are coordinates, and p, y are their conjugate momenta.

The variation of the variables p, q, y, x is described by the differential equations

$$\dot{p} = -\frac{\partial E}{\partial q}, \quad \dot{q} = \frac{\partial E}{\partial p}, \quad \dot{y} = -\varepsilon \frac{\partial E}{\partial x}, \quad \dot{x} = \varepsilon \frac{\partial E}{\partial y}.$$

*School of Mathematics, Loughborough University, UK, LE11 3TU

The variables p, q are called fast, and y, x are called slow variables. This system is called a slow-fast Hamiltonian system. The system with one degree of freedom in which $y = \text{const}$, $x = \text{const}$ is called unperturbed or fast system. Assume that when $y, x = \text{const}$, the phase portrait of the unperturbed system contains a domain filled by closed trajectories [1]:

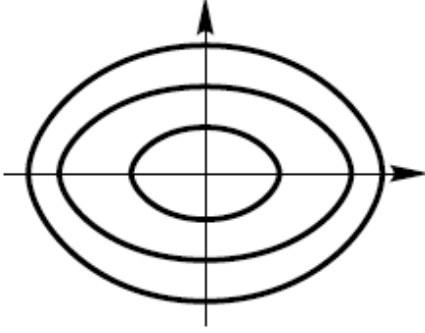


Figure 1

Here the frequency of motion is non-zero. So we can introduce action-angle variables

$$I = I(p, q, y, x), \quad \varphi = \varphi(p, q, y, x) \mod 2\pi$$

The action I represents the area surrounded by each trajectory. It is equal to this area divided by 2π . Denote $H_0(I, y, x)$ the Hamiltonian E expressed in terms of I, y, x . The approximation in which $I = \text{const}$ and dynamics of y, x is described by Hamiltonian system with Hamiltonian $H_0(I, y, x)$ is called an adiabatic approximation. In this approximation,

$$E(p, q, y, x) \equiv H_0(I, y, x),$$

$$\dot{y} = -\varepsilon \frac{\partial H_0(I, y, x)}{\partial x}, \quad \dot{x} = \varepsilon \frac{\partial H_0(I, y, x)}{\partial y}.$$

We will assume that in the phase portrait of this system there is a domain in which along trajectories $|x| \rightarrow \infty, y \rightarrow \text{const}$ as time $t \rightarrow \pm\infty$. Thus trajectories have a form shown in one of figures below:

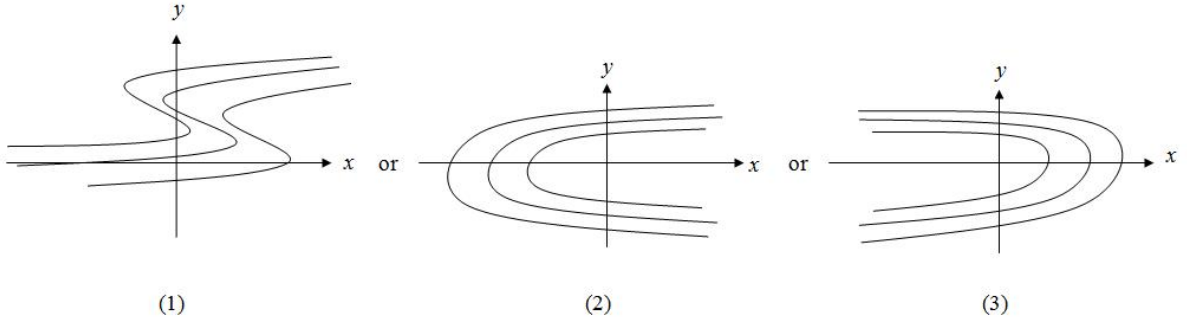


Figure 2

We consider the case when the adiabatic invariant of the action I along a trajectory of the original system possesses limiting values I_{\pm} as $t \rightarrow \pm\infty$. Their difference $\Delta I = I_+ - I_-$ is called an accuracy of persistence of the adiabatic invariant in slow-fast systems.

If this system is analytic, the value of ΔI is exponentially small [1]:

$$\Delta I = O(e^{-\frac{\gamma}{\varepsilon}}), \quad \gamma = \text{const} > 0.$$

The goal of this paper is to find an estimate of constant γ . After constructing some assumptions, we will then formulate and prove the conclusion.

2 Reduction to a standard form

For the original Hamiltonian $E(p, q, y, x)$ with $y, x = \text{const}$, q is the coordinate and p is conjugate momentum. In the unperturbed system, denote T the period of the trajectory and ω the frequency. Let $S(I, q, y, x)$ denote the generating function of canonical transformation $(p, q) \mapsto (I, \varphi)$. We have the relations [3]:

$$\varphi = \frac{\partial S(I, q, y, x)}{\partial I}, \quad p = \frac{\partial S(I, q, y, x)}{\partial q}.$$

For a fixed trajectory $E(p, q, y, x) = h = H_0(I, y, x)$, where h is a constant, we can express p as

$$p = P(h, q, y, x)$$

if $\frac{\partial E}{\partial p} \neq 0$. So the generating function has the following form [3]:

$$S(I, q, y, x) = \int_{q_0}^q P(h(I, y, x), q_1, y, x) dq_1,$$

where the constant q_0 is the initial value of q_1 .

Lemma 2.1. (see, e.g. [4])

$$\begin{aligned} \frac{\partial S}{\partial x} &= \int_0^t \left(\left\langle \frac{\partial E}{\partial x} \right\rangle - \frac{\partial E}{\partial x} \right) dt_1, \\ \frac{\partial S}{\partial y} &= \int_0^t \left(\left\langle \frac{\partial E}{\partial y} \right\rangle - \frac{\partial E}{\partial y} \right) dt_1 \end{aligned}$$

where

$$\left\langle \frac{\partial E}{\partial x} \right\rangle = \frac{1}{T} \oint \frac{\partial E}{\partial x} dt_1, \quad \left\langle \frac{\partial E}{\partial y} \right\rangle = \frac{1}{T} \oint \frac{\partial E}{\partial y} dt_1.$$

Now consider another canonical transformation $(p, q, y, x) \mapsto (\bar{I}, \bar{\varphi}, \bar{y}, \bar{x})$ with generating function $\varepsilon^{-1} \bar{y} x + S(\bar{I}, q, \bar{y}, x)$. The old conjugate variables are (p, q) and $(y, \varepsilon^{-1} x)$, and new pairs after transformation are $(\bar{I}, \bar{\varphi})$ and $(\bar{y}, \varepsilon^{-1} \bar{x})$.

Lemma 2.2. (see, e.g. [1])

$$y = \bar{y} + O(\varepsilon), \quad x = \bar{x} + O(\varepsilon), \quad I = \bar{I} + O(\varepsilon), \quad \varphi = \bar{\varphi} + O(\varepsilon)$$

and new Hamiltonian is

$$H = H_0(\bar{I}, \bar{y}, \bar{x}) + \varepsilon H_1(\bar{I}, \bar{\varphi}, \bar{y}, \bar{x}, \varepsilon).$$

We will consider the case that there exist limiting values of action variables I and \bar{I} as time tends to infinity. The conditions for this and the proof of existence of limits will be added later. Then we will see that the limiting values of I and \bar{I} are equal.

So far we have got a new Hamiltonian

$$H(\bar{I}, \bar{\varphi}, \bar{y}, \bar{x}, \varepsilon) = H_0(\bar{I}, \bar{y}, \bar{x}) + \varepsilon H_1(\bar{I}, \bar{\varphi}, \bar{y}, \bar{x}, \varepsilon).$$

For simplicity, omitting the bar symbols and dependence on ε , we obtain a Hamiltonian system in standard form:

$$H(I, \varphi, y, x) = H_0(I, y, x) + \varepsilon H_1(I, \varphi, y, x)$$

which has motion

$$\begin{aligned} \dot{I} &= -\varepsilon \frac{\partial H_1}{\partial \varphi}, \quad \dot{\varphi} = \frac{\partial H_0}{\partial I} + \varepsilon \frac{\partial H_1}{\partial I}, \\ \dot{y} &= -\varepsilon \left(\frac{\partial H_0}{\partial x} + \varepsilon \frac{\partial H_1}{\partial x} \right), \quad \dot{x} = \varepsilon \left(\frac{\partial H_0}{\partial y} + \varepsilon \frac{\partial H_1}{\partial y} \right). \end{aligned}$$

3 Statement of the problem

It is assumed, that in adiabatic approximation, $|x| \rightarrow \infty$, $y \rightarrow \text{const}$ as $t \rightarrow \pm\infty$ (see Figure 2). We will assume also, that when x, y are changing in accordance with adiabatic approximation, and p, q are fixed:

$$\frac{\partial E(p, q, x, y)}{\partial x} \rightarrow 0, \quad \text{as } t \rightarrow \pm\infty.$$

For definiteness we will consider the case when the trajectories in adiabatic approximation have the form shown in Figure 2(1). Result will be valid for the other two cases of Figure 2.

The function

$$\omega_0(I, y, x) = \frac{\partial H_0(I, y, x)}{\partial I}$$

is the frequency of unperturbed motion.

Let us assume that the following conditions are fulfilled.

Assumption 1°.

The functions H_0, H_1 can be analytically extended into a complex domain $D = D_I \times D_\varphi \times D_{xy}$, where D_I is a neighbourhood of a given real point I_* ,

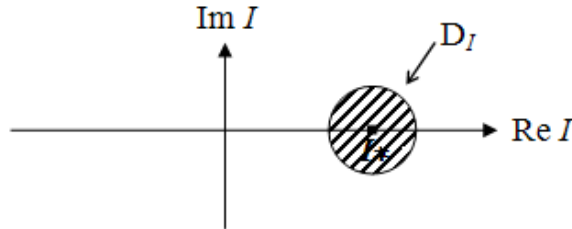


Figure 3

D_φ is a strip of a fixed width about the real axis,

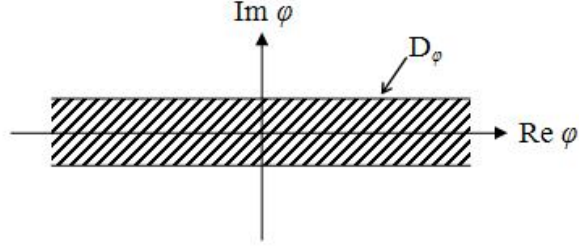


Figure 4

and D_{xy} is some domain in the complex plane \mathbb{C}^2 . The function $\omega_0(I, y, x)$ does not vanish in D and $|\omega_0| > \text{const}$. Function $H_0(I, y, x)$ satisfies

$$\left| \frac{\partial H_0}{\partial I} \right| < \text{const}, \quad \left| \frac{\partial H_0}{\partial y} \right| < \text{const}, \quad \left| \frac{\partial H_0}{\partial x} \right| < \text{const}, \quad \left(\frac{\partial H_0}{\partial y} \right)^2 + \left(\frac{\partial H_0}{\partial x} \right)^2 > \text{const}.$$

Now let us consider approximate Hamiltonian $H_0(I, y, x)$ which has motion

$$\dot{y} = -\varepsilon \frac{\partial H_0}{\partial x}, \quad \dot{x} = \varepsilon \frac{\partial H_0}{\partial y}.$$

For a fixed I_0 , we have $H_0(I_0, y, x) = h = \text{const}$. Assume that curves $H_0 = \text{const}$ are not closed:

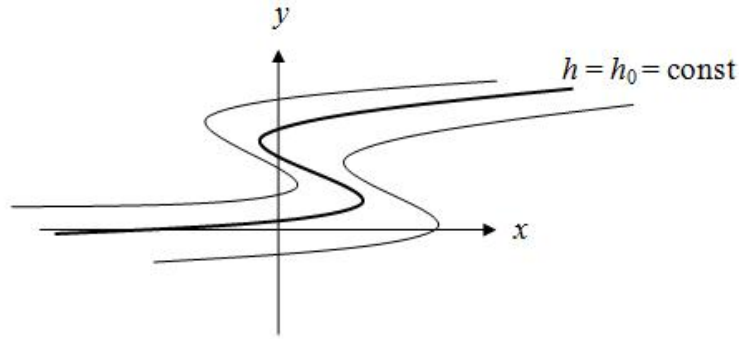


Figure 5

Introduce the slow time

$$\tau = \varepsilon t.$$

The differential equations of motion are

$$\frac{dy}{d\tau} = -\frac{\partial H_0}{\partial x}, \quad \frac{dx}{d\tau} = \frac{\partial H_0}{\partial y}.$$

Take $h = h_0$, h_0 is a point in some interval D_h with centre h_* , where h_* is a given real point:

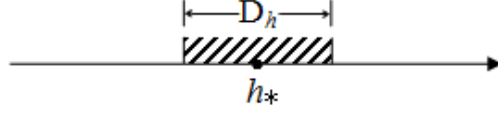


Figure 6

We can find the solution for describing the motion for approximate Hamiltonian H_0 with $H_0(I_0, y, x) = h_0$:

$$\begin{cases} y = Y(\tau, I_0, h_0) \\ x = X(\tau, I_0, h_0) \end{cases} \quad I_0 \in D_I, \quad h_0 \in D_h.$$

From now on we omit the dependence on h_0 .

Assumption 2°.

The solutions $Y(\tau, I_0)$, $X(\tau, I_0)$ can be analytically extended into a strip

$$D_\tau = \{\tau: |\operatorname{Im} \tau| < \sigma + \delta |\operatorname{Re} \tau|\}.$$

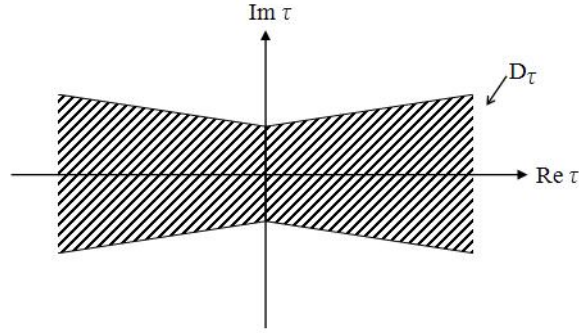


Figure 7

We suppose that for any initial point, the solution tends to infinity as time tends to infinity, i.e.

$$\lim_{\operatorname{Re} \tau \rightarrow \pm\infty} |X(\tau, I_0)| = \infty.$$

$H_0(I, y, x)$ satisfies

$$\lim_{\operatorname{Re} \tau \rightarrow \pm\infty} \frac{\partial H_0(I, Y(\tau, I_0), X(\tau, I_0))}{\partial x} = 0.$$

$H_1(I, \varphi, y, x)$ satisfies

$$|H_1(I, \varphi, Y(\tau, I_0), X(\tau, I_0))| < \frac{c}{1 + \left| \int_0^\tau \omega_0(I_0, Y(\tau_1, I_0), X(\tau_1, I_0)) d\tau_1 \right|^{2+\nu}}.$$

$\omega_0(I, y, x)$ satisfies

$$\operatorname{Im} \omega_0(I_0, Y(\tau, I_0), X(\tau, I_0)) \Rightarrow 0, \quad \text{as } \operatorname{Re} \tau \rightarrow \pm\infty, \operatorname{Im} I_0 \rightarrow 0.$$

Here σ, δ, c, ν are positive constants.

Lemma 3.1. *For $I \in \tilde{D}_I = D_I - \delta_I$, $\varphi \in \tilde{D}_\varphi = D_\varphi - \delta_\varphi$, and $\tau \in \tilde{D}_\tau = \{\tau: |\operatorname{Im} \tau| < \sigma - \delta_\tau + \delta|\operatorname{Re} \tau|\}$,*

$$\begin{aligned} \left| \frac{\partial H_1(I, \varphi, Y(\tau, I_0), X(\tau, I_0))}{\partial I} \right| &< \frac{\operatorname{const}}{1 + \left| \int_0^\tau \omega_0(I_0, Y(\tau_1, I_0), X(\tau_1, I_0)) d\tau_1 \right|^{2+\nu}}, \\ \left| \frac{\partial H_1(I, \varphi, Y(\tau, I_0), X(\tau, I_0))}{\partial \varphi} \right| &< \frac{\operatorname{const}}{1 + \left| \int_0^\tau \omega_0(I_0, Y(\tau_1, I_0), X(\tau_1, I_0)) d\tau_1 \right|^{2+\nu}}, \\ \left| \frac{\partial H_1(I, \varphi, Y(\tau, I_0), X(\tau, I_0))}{\partial \tau} \right| &< \frac{\operatorname{const}}{1 + \left| \int_0^\tau \omega_0(I_0, Y(\tau_1, I_0), X(\tau_1, I_0)) d\tau_1 \right|^{2+\nu}}, \end{aligned}$$

where $\delta_I, \delta_\varphi, \delta_\tau$ are positive constants.

Proof. $\operatorname{Im} \omega_0(I_0, Y(\tau, I_0), X(\tau, I_0)) \Rightarrow 0$, as $\operatorname{Re} \tau \rightarrow \pm\infty, \operatorname{Im} I_0 \rightarrow 0$
implies that $\forall \rho > 0, \exists \Sigma > 0, \Gamma > 0$, such that if $|\operatorname{Im} I_0| < \Sigma, |\operatorname{Re} \tau| > \Gamma$, then

$$|\operatorname{Im} \omega_0| < \rho.$$

Since we know that $|\omega_0| > c_1 = \operatorname{const}$, assume $|\operatorname{Im} \omega_0| < \frac{c_1}{10}$ for $|\operatorname{Im} I_0| < \Sigma, |\operatorname{Re} \tau| > \Gamma$, and therefore,

$$|\operatorname{Re} \omega_0| > \frac{9}{10}c_1.$$

Let $\tau = \lambda + i\mu$. The integration $\left| \int_0^\tau \omega_0(I_0, Y(\tau_1, I_0), X(\tau_1, I_0)) d\tau_1 \right|$ does not depend on the path.

For $|\operatorname{Re} \tau| > \max \left\{ \Gamma, 2 \left(\frac{9}{10} c_1 - \rho \right)^{-1} \left(\frac{9}{10} c_1 + \rho + a + b \right) \Gamma \right\}$, where a, b are positive constants satisfying

$$\int_0^\Gamma |\operatorname{Re} \omega_0| d\lambda \leq a \cdot \Gamma, \quad \int_0^\Gamma |\operatorname{Im} \omega_0| d\lambda \leq b \cdot \Gamma,$$

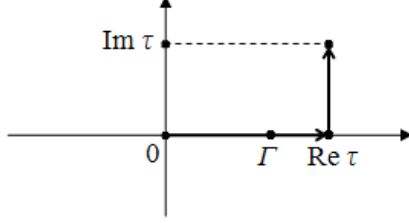


Figure 8

we have

$$\begin{aligned} \left| \int_0^\tau \omega_0 d\tau_1 \right| &= \left| \int_0^{\operatorname{Re} \tau} \omega_0 d\tau_1 + \int_{\operatorname{Re} \tau}^\tau \omega_0 d\tau_1 \right| \\ &= \left| \int_0^{\operatorname{Re} \tau} \omega_0 d(\lambda + i\mu) + \int_{\operatorname{Re} \tau}^\tau \omega_0 d(\lambda + i\mu) \right| \\ &= \left| \int_0^{\operatorname{Re} \tau} \omega_0 d\lambda + \int_{\operatorname{Re} \tau}^\tau \omega_0 d(i\mu) \right| \\ &= \left| \int_0^{\operatorname{Re} \tau} \operatorname{Re} \omega_0 d\lambda + \int_0^{\operatorname{Re} \tau} i \operatorname{Im} \omega_0 d\lambda + \int_{\operatorname{Re} \tau}^\tau \operatorname{Re} \omega_0 d(i\mu) + \int_{\operatorname{Re} \tau}^\tau i \operatorname{Im} \omega_0 d(i\mu) \right| \\ &= \left| \int_0^\Gamma \operatorname{Re} \omega_0 d\lambda + \int_\Gamma^{\operatorname{Re} \tau} \operatorname{Re} \omega_0 d\lambda + \int_0^\Gamma i \operatorname{Im} \omega_0 d\lambda + \int_\Gamma^{\operatorname{Re} \tau} i \operatorname{Im} \omega_0 d\lambda \right. \\ &\quad \left. + \int_{\operatorname{Re} \tau}^\tau \operatorname{Re} \omega_0 d(i\mu) + \int_{\operatorname{Re} \tau}^\tau i \operatorname{Im} \omega_0 d(i\mu) \right| \end{aligned}$$

$$\begin{aligned}
&\geq \left\| \left| \int_{\Gamma}^{\operatorname{Re} \tau} \operatorname{Re} \omega_0 \, d\lambda + \int_{\operatorname{Re} \tau}^{\tau} \operatorname{Re} \omega_0 \, d(i\mu) \right| - \left| \int_{\Gamma}^{\operatorname{Re} \tau} |\operatorname{Im} \omega_0| \, d\lambda + \int_{\operatorname{Re} \tau}^{\tau} |\operatorname{Im} \omega_0| \, d(i\mu) \right. \right. \\
&\quad \left. \left. + \int_0^{\Gamma} |\operatorname{Re} \omega_0| \, d\lambda + \int_0^{\Gamma} |\operatorname{Im} \omega_0| \, d\lambda \right| \right\| \\
&> \left| \frac{9}{10} c_1 \cdot (\operatorname{Re} \tau - \Gamma) + \frac{9}{10} c_1 \cdot i \operatorname{Im} \tau \right| - \left| \rho \cdot (\operatorname{Re} \tau - \Gamma) + \rho \cdot i \operatorname{Im} \tau + a\Gamma + b\Gamma \right| \\
&= \left| \frac{9}{10} c_1 \cdot \tau - \frac{9}{10} c_1 \cdot \Gamma \right| - \left| \rho \cdot \tau - \rho \cdot \Gamma + a\Gamma + b\Gamma \right| \\
&\geq \left| \left(\frac{9}{10} c_1 |\tau| - \frac{9}{10} c_1 \Gamma \right) - \left(\rho |\tau| + \rho \Gamma + a\Gamma + b\Gamma \right) \right| \\
&= \left| \left(\frac{9}{10} c_1 - \rho \right) |\tau| - \left(\frac{9}{10} c_1 + \rho + a + b \right) \Gamma \right| \\
&> \left(\frac{9}{10} c_1 - \rho \right) |\tau|
\end{aligned}$$

as $\left(\frac{9}{10} c_1 - \rho \right) |\tau| > 2 \left(\frac{9}{10} c_1 + \rho + a + b \right) \Gamma$.

We can find a positive constant $c_2 < \frac{9}{10} c_1 - \rho$, such that

$$\left| \int_0^{\tau} \omega_0(I_0, Y(\tau_1, I_0), X(\tau_1, I_0)) \, d\tau_1 \right| > \left(\frac{9}{10} c_1 - \rho \right) |\tau| > c_2 |\tau|.$$

For $|\operatorname{Re} \tau| \leq \max \left\{ \Gamma, 2 \left(\frac{9}{10} c_1 - \rho \right)^{-1} \left(\frac{9}{10} c_1 + \rho + a + b \right) \Gamma \right\}$, it is evident that we can find another positive constant c_3 such that

$$\left| \int_0^{\tau} \omega_0(I_0, Y(\tau_1, I_0), X(\tau_1, I_0)) \, d\tau_1 \right| > c_3 |\tau|.$$

So

$$\begin{aligned}
|H_1(I, \varphi, Y(\tau, I_0), X(\tau, I_0))| &< \frac{c}{1 + \left| \int_0^{\tau} \omega_0(I_0, Y(\tau_1, I_0), X(\tau_1, I_0)) \, d\tau_1 \right|^{2+\nu}} \\
&< \frac{c}{1 + (\operatorname{const} \cdot |\tau|)^{2+\nu}} \\
&< \frac{\operatorname{const}}{1 + |\tau|^{2+\nu}}.
\end{aligned}$$

From Cauchy estimate [5], we can easily get that, in \tilde{D}_I and \tilde{D}_φ ,

$$\left| \frac{\partial H_1(I, \varphi, Y(\tau, I_0), X(\tau, I_0))}{\partial I} \right| < \frac{\text{const}}{1 + |\tau|^{2+\nu}},$$

$$\left| \frac{\partial H_1(I, \varphi, Y(\tau, I_0), X(\tau, I_0))}{\partial \varphi} \right| < \frac{\text{const}}{1 + |\tau|^{2+\nu}}.$$

Now let us prove $\left| \frac{\partial H_1(I, \varphi, Y(\tau, I_0), X(\tau, I_0))}{\partial \tau} \right| < \frac{\text{const}}{1 + |\tau|^{2+\nu}}$. From

$$D_\tau = \{\tau: |\text{Im } \tau| < \sigma + \delta|\text{Re } \tau|\},$$

for simplicity, assume $\delta_\tau = \frac{\sigma}{10}$, and then

$$\tilde{D}_\tau = \{\tau: |\text{Im } \tau| < \sigma - \frac{\sigma}{10} + \delta|\text{Re } \tau|\}.$$

In the following figure, mark the boundary of D_τ with solid lines, and that of \tilde{D}_τ with dashed lines:

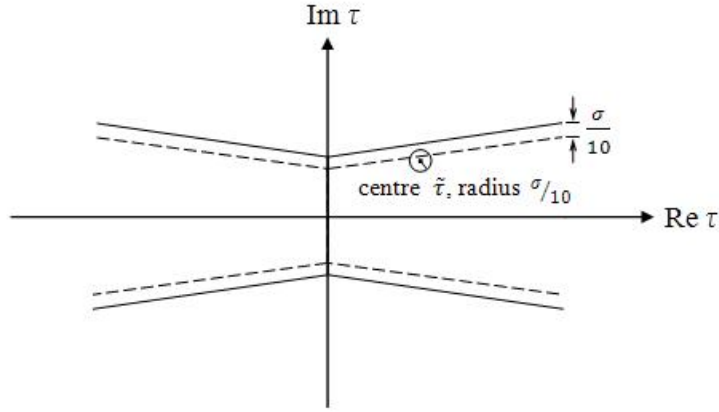


Figure 9

For $\tilde{\tau} \in \tilde{D}_\tau$, the estimate of $|H_1|$ on the circle of radius $\frac{\sigma}{10}$ around $\tilde{\tau}$ is

$$|H_1| < \frac{\text{const}}{1 + \min_{\theta} \left| \tilde{\tau} + \frac{\sigma}{10} e^{i\theta} \right|^{2+\nu}}. \text{ If } |\tilde{\tau}| > \sigma, \text{ then}$$

$$|H_1| < \frac{\text{const}}{1 + \min_{\theta} \left| \tilde{\tau} + \frac{\sigma}{10} e^{i\theta} \right|^{2+\nu}} < \frac{\text{const}}{1 + \left| \frac{9}{10} \tilde{\tau} \right|^{2+\nu}}.$$

So, because of the Cauchy estimate

$$\left\| \frac{\partial H_1}{\partial \tau} \right\|_{\tilde{\tau}} < \frac{\text{const}}{1 + \left| \frac{9}{10} \tilde{\tau} \right|^{2+\nu}} \bigg/ \frac{\sigma}{10} < \frac{\text{const}}{1 + |\tilde{\tau}|^{2+\nu}}.$$

If $|\tilde{\tau}| \leq \sigma$, then because of the Cauchy estimate,

$$\left\| \frac{\partial H_1}{\partial \tau} \right\|_{\tilde{\tau}} < \text{const} \bigg/ \frac{\sigma}{10} < \frac{\text{const}}{1 + |\tilde{\tau}|^{2+\nu}}.$$

Therefore,

$$\left| \frac{\partial H_1(I, \varphi, Y(\tau, I_0), X(\tau, I_0))}{\partial \tau} \right| < \frac{\text{const}}{1 + |\tau|^{2+\nu}}.$$

We can find a constant c_4 such that

$$\left| \int_0^\tau \omega_0(I_0, Y(\tau_1, I_0), X(\tau_1, I_0)) \, d\tau_1 \right| < c_4 |\tau|.$$

Then

$$\begin{aligned} \frac{\text{const}}{1 + |\tau|^{2+\nu}} &< \frac{\text{const}}{1 + \left(c_4^{-1} \left| \int_0^\tau \omega_0(I_0, Y(\tau_1, I_0), X(\tau_1, I_0)) \, d\tau_1 \right| \right)^{2+\nu}} \\ &< \frac{\text{const}}{1 + \left| \int_0^\tau \omega_0(I_0, Y(\tau_1, I_0), X(\tau_1, I_0)) \, d\tau_1 \right|^{2+\nu}}. \end{aligned}$$

Therefore, for $I \in \tilde{D}_I$, $\varphi \in \tilde{D}_\varphi$, $\tau \in \tilde{D}_\tau$,

$$\begin{aligned} \left| \frac{\partial H_1(I, \varphi, Y(\tau, I_0), X(\tau, I_0))}{\partial I} \right| &< \frac{\text{const}}{1 + \left| \int_0^\tau \omega_0(I_0, Y(\tau_1, I_0), X(\tau_1, I_0)) \, d\tau_1 \right|^{2+\nu}}, \\ \left| \frac{\partial H_1(I, \varphi, Y(\tau, I_0), X(\tau, I_0))}{\partial \varphi} \right| &< \frac{\text{const}}{1 + \left| \int_0^\tau \omega_0(I_0, Y(\tau_1, I_0), X(\tau_1, I_0)) \, d\tau_1 \right|^{2+\nu}}, \\ \left| \frac{\partial H_1(I, \varphi, Y(\tau, I_0), X(\tau, I_0))}{\partial \tau} \right| &< \frac{\text{const}}{1 + \left| \int_0^\tau \omega_0(I_0, Y(\tau_1, I_0), X(\tau_1, I_0)) \, d\tau_1 \right|^{2+\nu}}. \end{aligned}$$

□

Assumption 3°.

The level lines

$$\operatorname{Im} \int_0^\tau \omega_0(I, Y(\tau_1, I), X(\tau_1, I)) \, d\tau_1 = B = \text{const}, \quad 0 \leq |B| \leq \gamma, \quad I \in D_I$$

lie in the domain D_τ and have a positive distance from the boundary of D_τ .

Now consider the exact solution $I(t)$, $\varphi(t)$, $y(t)$, $x(t)$ of the Hamiltonian system with Hamiltonian

$$H(I, \varphi, y, x) = H_0(I, y, x) + \varepsilon H_1(I, \varphi, y, x)$$

with real initial conditions $I(0)$, $\varphi(0)$, $y(0)$, $x(0)$ in $D = D_I \times D_\varphi \times D_{xy}$ at $t = \tau/\varepsilon = 0$, and $H(I(0), \varphi(0), y(0), x(0)) = h_0$. The adiabatic invariant of the action $I(t)$ satisfies

$$\frac{dI}{dt} = -\varepsilon \frac{\partial H_1(I, \varphi, y, x)}{\partial \varphi}$$

and thus

$$I(t) = I(0) - \varepsilon \int_0^t \frac{\partial H_1(I(t_1), \varphi(t_1), y(t_1), x(t_1))}{\partial \varphi} \, dt_1.$$

Theorem 3.1. *There exist limiting values*

$$I_\pm = \lim_{t \rightarrow \pm\infty} I(t)$$

and their difference $\Delta I = I_+ - I_-$ satisfies the estimate

$$\Delta I = O(e^{-\frac{\gamma}{\varepsilon}}), \quad \gamma = \text{const} > 0$$

with the constant γ introduced in Assumption 3°.

Remark. The method of continuous averaging [8] gives the same estimate for γ , as the above-stated theorem.

4 Example

Paper [7] gives us several examples of slow-fast Hamiltonian systems, in one of which the Hamiltonian is as follows:

$$H(I, \varphi, y, x) = \omega I + \frac{y^2}{2} + V_0(x) + \varepsilon g(\varphi) V_1(x).$$

Here y, x are slow variables and I, φ are fast ones. We explicitly treat the case of $V_0(x) = e^{-x}$. Concerning V_1 , we need it to be real analytic and tend to zero as $x \rightarrow +\infty$. For simplicity we choose $V_1 \equiv e^{-x}$. The unperturbed Hamiltonian system is

$$H_0(I, y, x) = \omega I + \frac{y^2}{2} + V_0(x) = \omega I + \frac{y^2}{2} + e^{-x}.$$

Let $H_0(I, y, x) = \omega I + \frac{y^2}{2} + e^{-x} = h_0 = \text{const}$ be the total energy of the unperturbed system. We can draw the phase portrait of the system to describe the motion:

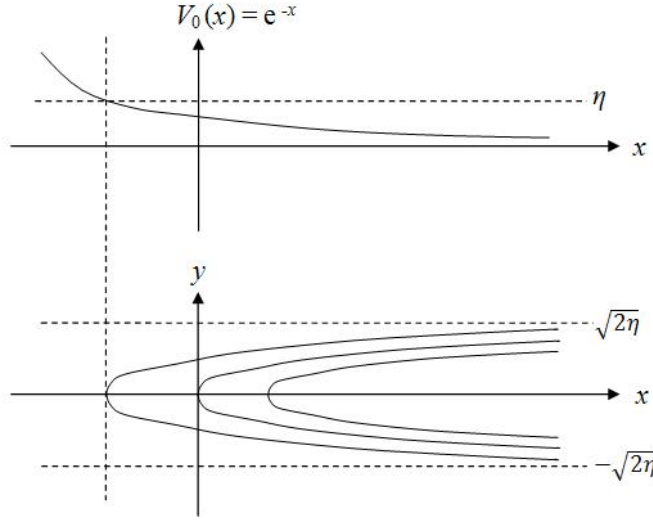


Figure 10

Take $\tilde{\xi}$ as the slow time of motion. From the Hamiltonian $H_0(I, y, x) = \omega I + \frac{y^2}{2} + e^{-x} = h_0$, we know that

$$\frac{dx}{d\tilde{\xi}} = \frac{\partial H_0}{\partial y} = y.$$

So

$$\frac{1}{2} \left(\frac{dx}{d\tilde{\xi}} \right)^2 + e^{-x} = h_0 - \omega I = \tilde{\eta}.$$

Thus we can express x and y via $(\tilde{\xi}, \tilde{\eta})$:

$$x = x(\tilde{\xi}, \tilde{\eta}), \quad y = y(\tilde{\xi}, \tilde{\eta})$$

with initial data $x^0 = x(0, \tilde{\eta})$ and $y^0 = \sqrt{2(\tilde{\eta} - e^{-x^0})}$.

After solving differential equations, we can obtain the solutions as (see [7])

$$x(\tilde{\xi}, \tilde{\eta}) = \log \left(\frac{1}{\tilde{\eta}} \left(\cosh \sqrt{\tilde{\eta}/2} (\tilde{\xi} - \tilde{\xi}^0) \right)^2 \right), \quad y(\tilde{\xi}, \tilde{\eta}) = \sqrt{2\tilde{\eta}} \tanh \sqrt{\tilde{\eta}/2} (\tilde{\xi} - \tilde{\xi}^0).$$

For simplicity we take the initial value of $\tilde{\xi}$ as $\tilde{\xi}^0 = 0$. So $x^0 = x(0, \tilde{\eta}) = \log \frac{1}{\tilde{\eta}}$, $y^0 = 0$.

Now after the canonical transformation of variables from (x, y) to $(\tilde{\xi}, \tilde{\eta})$, the Hamiltonian H_0 becomes $K_0(\tilde{\xi}, \tilde{\eta}) = \omega I + \tilde{\eta}$ and the new Hamiltonian is

$$K(\tilde{\xi}, \tilde{\eta}, I, \varphi) = \omega I + \tilde{\eta} + \varepsilon f(\tilde{\xi}, \tilde{\eta})g(\varphi)$$

with

$$f(\tilde{\xi}, \tilde{\eta}) = V_1(x(\tilde{\xi}, \tilde{\eta})) = e^{-\log\left(\frac{1}{\tilde{\eta}}\left(\cosh\sqrt{\tilde{\eta}/2}\tilde{\xi}\right)^2\right)} = \frac{\tilde{\eta}}{\left(\cosh\sqrt{\tilde{\eta}/2}\tilde{\xi}\right)^2}.$$

The differential equations for describing the motion are

$$\begin{aligned}\dot{\tilde{\xi}} &= \varepsilon + \varepsilon^2 \frac{\partial f(\tilde{\xi}, \tilde{\eta})}{\partial \tilde{\eta}} g(\varphi), \\ \dot{\tilde{\eta}} &= -\varepsilon^2 \frac{\partial f(\tilde{\xi}, \tilde{\eta})}{\partial \tilde{\xi}} g(\varphi), \\ \dot{I} &= -\varepsilon f(\tilde{\xi}, \tilde{\eta}) \frac{dg(\varphi)}{d\varphi}, \\ \dot{\varphi} &= \omega.\end{aligned}$$

For $g(\varphi)$, we suppose that $g(\varphi)$ is analytic in the strip $|\operatorname{Im} \varphi| < \rho$ with constant $\rho > 0$, and bounded:

$$|g(\varphi)| \leq 1, \quad \text{for } |\operatorname{Im} \varphi| < \rho.$$

For $f(\tilde{\xi}, \tilde{\eta})$, we can easily prove that

$$|f(\tilde{\xi}, \tilde{\eta})| < \frac{\text{const}}{1 + |\tilde{\xi}|^{2+\nu}},$$

and

$$\begin{aligned}\left| \frac{\partial f(\tilde{\xi}, \tilde{\eta})}{\partial \tilde{\xi}} \right| &< \frac{\text{const}}{1 + |\tilde{\xi}|^{2+\nu}}, \\ \left| \frac{\partial f(\tilde{\xi}, \tilde{\eta})}{\partial \tilde{\eta}} \right| &< \frac{\text{const}}{1 + |\tilde{\xi}|^{2+\nu}}.\end{aligned}$$

Take D_I as a neighbourhood of a given real point I_* , and D_φ is a strip $|\operatorname{Im} \varphi| < \rho$. Now let us determine the domain of $\tilde{\xi}$ by finding singularities of $f(\tilde{\xi}, \tilde{\eta})$:

$$f(\tilde{\xi}, \tilde{\eta}) = \frac{\tilde{\eta}}{\left(\cosh\sqrt{\tilde{\eta}/2}\tilde{\xi}\right)^2} = \frac{4\tilde{\eta}}{\left(e^{\sqrt{\tilde{\eta}/2}\tilde{\xi}} + e^{-\sqrt{\tilde{\eta}/2}\tilde{\xi}}\right)^2}.$$

Points of singularities should satisfy

$$\begin{aligned}
e^{\sqrt{\tilde{\eta}/2}\tilde{\xi}} + e^{-\sqrt{\tilde{\eta}/2}\tilde{\xi}} &= 0 \\
e^{2\sqrt{\tilde{\eta}/2}\tilde{\xi}} + 1 &= 0 \\
e^{\sqrt{2\tilde{\eta}}\tilde{\xi}} &= -1 \\
e^{\operatorname{Re}\sqrt{2\tilde{\eta}}\tilde{\xi}} \left(\cos \operatorname{Im}\sqrt{2\tilde{\eta}}\tilde{\xi} + i \sin \operatorname{Im}\sqrt{2\tilde{\eta}}\tilde{\xi} \right) &= -1
\end{aligned}$$

$$\begin{aligned}
\text{So} \quad \cos \operatorname{Im}\sqrt{2\tilde{\eta}}\tilde{\xi} &= -e^{-\operatorname{Re}\sqrt{2\tilde{\eta}}\tilde{\xi}} < 0, \\
\sin \operatorname{Im}\sqrt{2\tilde{\eta}}\tilde{\xi} &= 0.
\end{aligned}$$

$$\begin{aligned}
\text{Therefore,} \quad \operatorname{Im}\sqrt{2\tilde{\eta}}\tilde{\xi} &= (2k+1)\pi, \quad k \in \mathbb{Z}, \\
\text{and} \quad \operatorname{Re}\sqrt{2\tilde{\eta}}\tilde{\xi} &= 0.
\end{aligned}$$

Thus

$$\sqrt{2\tilde{\eta}}\tilde{\xi} = (2k+1)\pi i, \quad k \in \mathbb{Z}.$$

The domain of $\tilde{\eta}$ denoted as $D_{\tilde{\eta}}$ is a complex ball around a real value $\tilde{\eta}_*$. Any value $\tilde{\eta}^0 \in D_{\tilde{\eta}}$ can be taken to determine the singularities. Therefore, we obtain the singularities as

$$\tilde{\xi} = (2k+1) \frac{\pi}{\sqrt{2\tilde{\eta}^0}} i, \quad k \in \mathbb{Z}.$$

From the above conclusion, if we take $\tilde{\xi}$ in the strip

$$D_{\tilde{\xi}} = \{\tilde{\xi}: |\operatorname{Im}\tilde{\xi}| < \frac{\pi}{\sqrt{2\tilde{\eta}^0}} + \delta |\operatorname{Re}\tilde{\xi}|\},$$

where δ is a positive constant, $f(\tilde{\xi}, \tilde{\eta})$ is analytic for any $\tilde{\eta} \in D_{\tilde{\eta}}$.

The following figure shows the boundary of domain $D_{\tilde{\xi}}$:

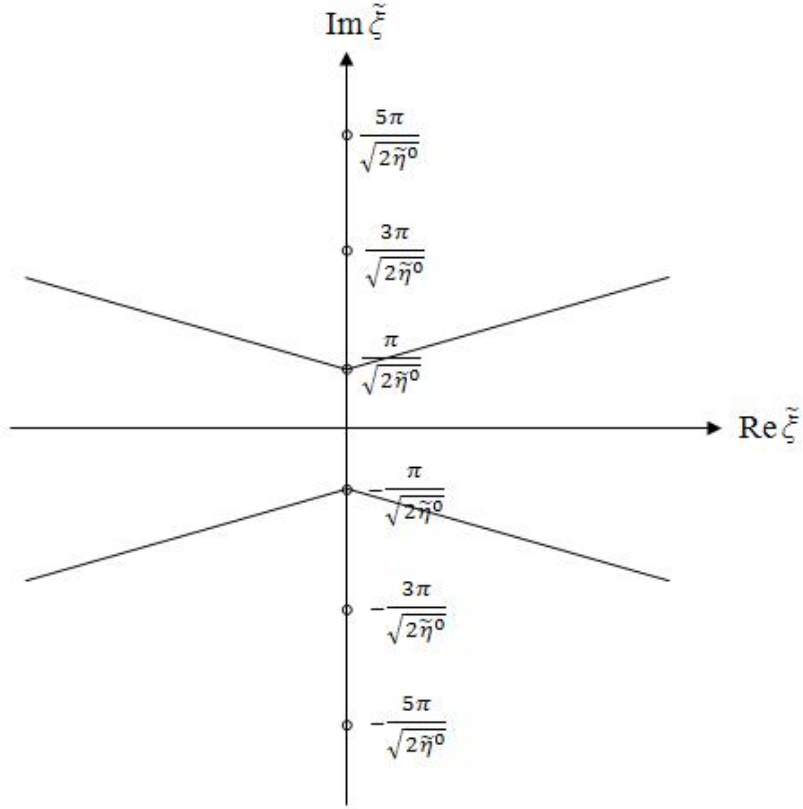


Figure 11

We can find the level lines

$$\text{Im} \int_0^{\tilde{\xi}} \omega d\tilde{\xi}_1 = \omega \text{Im} \tilde{\xi} = B = \text{const}, \quad 0 \leq |B| \leq \gamma$$

lying in the domain $D_{\tilde{\xi}}$, which have a positive distance from the boundary of this domain. So γ has been found as any constant such that

$$\gamma < \frac{\pi\omega}{\sqrt{2\tilde{\eta}^0}}, \quad \text{for such definition of } \tilde{\eta}_0.$$

According to Theorem 1, the accuracy of conservation of adiabatic invariant ΔI has the estimate:

$$\Delta I = O(e^{-\frac{\gamma}{\varepsilon}}).$$

This estimate coincides with that in [7], where it was obtained by a different method.

5 Proof of the Theorem

The original system is described by Hamiltonian

$$H(I, \varphi, y, x) = H_0(I, y, x) + \varepsilon H_1(I, \varphi, y, x)$$

with motion in time t :

$$\begin{aligned} \dot{I} &= -\varepsilon \frac{\partial H_1}{\partial \varphi}, \quad \dot{\varphi} = \frac{\partial H_0}{\partial I} + \varepsilon \frac{\partial H_1}{\partial I}, \\ \dot{y} &= -\varepsilon \left(\frac{\partial H_0}{\partial x} + \varepsilon \frac{\partial H_1}{\partial x} \right), \quad \dot{x} = \varepsilon \left(\frac{\partial H_0}{\partial y} + \varepsilon \frac{\partial H_1}{\partial y} \right). \end{aligned}$$

As the frequency $\omega_0(I, y, x) = \frac{\partial H_0(I, y, x)}{\partial I}$ does not vanish in our domain D , we can express I from H via φ, y, x and a constant of energy. This is the procedure of isoenergetic reduction on the energy level $H(I, \varphi, y, x) = h_0$, $h_0 = \text{const}$:

$$-I(y, x, \varphi, h_0) = F_0(y, x, h_0) + \varepsilon F_1(y, x, \varphi, h_0).$$

Here $(-I)$ is the new Hamiltonian, φ is the new time, and (y, x) is in a small neighbourhood of \tilde{D}_{xy} , which can be written as $\tilde{D}_{xy} + \delta_{xy}$, where

$$\tilde{D}_{xy} = \left\{ (y, x) : \begin{cases} y = Y(\tau, I_0, h_0) \\ x = X(\tau, I_0, h_0) \end{cases}, \tau \in D_\tau, I_0 \in D_I \right\},$$

and δ_{xy} is some positive constant.

We do not indicate dependence of Y, X on ε . The differential equations of motion are

$$\frac{dy}{d\varphi} = -\varepsilon \left(\frac{\partial F_0}{\partial x} + \varepsilon \frac{\partial F_1}{\partial x} \right), \quad \frac{dx}{d\varphi} = \varepsilon \left(\frac{\partial F_0}{\partial y} + \varepsilon \frac{\partial F_1}{\partial y} \right).$$

Since we know functions H_0 and H_1 are analytic in D , from the implicit function theorem, functions F_0 and F_1 are also analytic in $D_\varphi \times (\tilde{D}_{xy} + \delta_{xy})$.

Now consider the approximate system with Hamiltonian

$$F_0(y, x, h_0) = -I_0.$$

Introduce slow time $\xi = \varepsilon \varphi$, and the motion is

$$\frac{dy}{d\xi} = -\frac{\partial F_0}{\partial x}, \quad \frac{dx}{d\xi} = \frac{\partial F_0}{\partial y}.$$

We can find the solution for the above approximate system:

$$\begin{cases} y = \hat{Y}(\xi, I_0, h_0) \\ x = \hat{X}(\xi, I_0, h_0) \end{cases} \quad I_0 \in D_I, \quad h_0 \in D_h.$$

In what follows, we do not indicate the dependence on constant h_0 . The relation between solutions $Y(\tau, I_0)$, $X(\tau, I_0)$ and $\hat{Y}(\xi, I_0)$, $\hat{X}(\xi, I_0)$ follows from the formula

$$\xi = \int_0^\tau \omega_0(I_0, Y(\tau_1, I_0), X(\tau_1, I_0)) d\tau_1.$$

Solution $\hat{Y}(\xi, I_0)$, $\hat{X}(\xi, I_0)$ can be analytically extended into some domain D_ξ with respect to ξ . The domain D_ξ will be introduced in later lemma. As we know,

$$\text{Im } \omega_0(I_0, Y(\tau, I_0), X(\tau, I_0)) \Rightarrow 0, \quad \text{as } \text{Re } \tau \rightarrow \pm\infty, \text{ Im } I_0 \rightarrow 0.$$

It implies that ω_0 uniformly tends to a real function depending on $\text{Re } I_0$ as time tends to infinity and I_0 tends to real. Let us denote the limits as $\omega_\pm(\text{Re } I_0)$:

$$\omega_0(I_0, Y(\tau_1, I_0), X(\tau_1, I_0)) \Rightarrow \omega_\pm(\text{Re } I_0), \quad \text{as } \text{Re } \tau \rightarrow \pm\infty, \text{ Im } I_0 \rightarrow 0.$$

Lemma 5.1. *For $I_0 \in D_I$ close to the imaginary axis, the image of the domain D_τ under the map $\tau \mapsto \xi$ given by the formula*

$$\xi = \int_0^\tau \omega_0(I_0, Y(\tau_1, I_0), X(\tau_1, I_0)) d\tau_1$$

is a strip around the real axis that contains a domain

$$D_\xi = \left\{ \xi : \begin{cases} |\text{Im } \xi| < \sigma_1 + \delta_1 |\text{Re } \xi|, & |\text{Re } \xi| > c_5 \Gamma \\ |\text{Im } \xi| < \sigma_2, & |\text{Re } \xi| \leq c_5 \Gamma \end{cases} \right\}$$

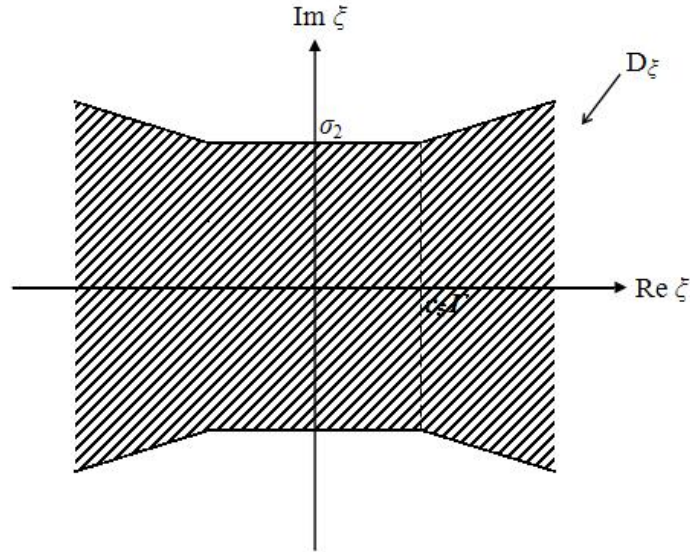


Figure 12

where $\sigma_1, \sigma_2, \delta_1, c_5$ are positive constants and $\sigma_2 = \sigma_1 + \delta_1 \cdot c_5 \Gamma$ for continuity. Straight lines $\text{Im } \xi = \pm \gamma$ lie in D_ξ and have a positive distance from the boundary of D_ξ (i.e. $\gamma < \sigma_2$).

Lemma 5.2. *The function $F_1(y, x, \varphi)$ satisfies the estimate:*

$$|F_1(\widehat{Y}(\xi, I_0), \widehat{X}(\xi, I_0), \varphi)| < \frac{\text{const}}{1 + |\xi|^{2+\nu}}.$$

Proof. First of all let us list two forms of Hamiltonian:

$$\begin{aligned} H_0(I, y, x) + \varepsilon H_1(I, \varphi, y, x) &= h_0, \\ -I &= F_0(y, x) + \varepsilon F_1(y, x, \varphi). \end{aligned}$$

Therefore, we have

$$H_0(-(F_0 + \varepsilon F_1), y, x) + \varepsilon H_1(-(F_0 + \varepsilon F_1), \varphi, y, x) = h_0.$$

$$\text{Then let } \varepsilon = 0 : \quad H_0(-F_0, y, x) = h_0.$$

So

$$\begin{aligned} H_0(-(F_0 + \varepsilon F_1), y, x) + \varepsilon H_1(-(F_0 + \varepsilon F_1), \varphi, y, x) &= H_0(-F_0, y, x), \\ H_0(-(F_0 + \varepsilon F_1), y, x) - H_0(-F_0, y, x) &= -\varepsilon H_1(-(F_0 + \varepsilon F_1), \varphi, y, x). \end{aligned}$$

Applying the intermediate value theorem here, we get that there exist some value \tilde{I} , such that

$$\begin{aligned} \frac{\partial H_0(\tilde{I}, y, x)}{\partial I} \cdot (-\varepsilon F_1) &= -\varepsilon H_1(-(F_0 + \varepsilon F_1), \varphi, y, x) \\ \text{so } F_1(y, x, \varphi) &= \frac{1}{\frac{\partial H_0(\tilde{I}, y, x)}{\partial I}} \cdot H_1(I, \varphi, y, x) \end{aligned}$$

We know from assumptions that

$$|\omega_0| = \left| \frac{\partial H_0}{\partial I} \right| > \text{const}$$

$$\text{and } |H_1(I, \varphi, Y(\tau, I_0), X(\tau, I_0))| < \frac{c}{1 + \left| \int_0^\tau \omega_0(I_0, Y(\tau_1, I_0), X(\tau_1, I_0)) d\tau_1 \right|^{2+\nu}}$$

Therefore,

$$\begin{aligned} |F_1(\widehat{Y}(\xi, I_0), \widehat{X}(\xi, I_0), \varphi)| &< \text{const} \cdot |H_1(I, \varphi, Y(\tau, I_0), X(\tau, I_0))| \\ &< \frac{\text{const}}{1 + |\xi|^{2+\nu}}. \end{aligned}$$

□

Corollary 5.1. For $\varphi \in \tilde{D}_\varphi = D_\varphi - \delta_\varphi$, $\xi \in \tilde{D}_\xi = D_\xi - \delta_\xi$,

$$\left| \frac{\partial F_1(\hat{Y}(\xi, I_0), \hat{X}(\xi, I_0), \varphi, h_0)}{\partial \varphi} \right| < \frac{\text{const}}{1 + |\xi|^{2+\nu}},$$

$$\left| \frac{\partial F_1(\hat{Y}(\xi, I_0), \hat{X}(\xi, I_0), \varphi, h_0)}{\partial \xi} \right| < \frac{\text{const}}{1 + |\xi|^{2+\nu}},$$

where δ_ξ is a positive constant.

Now let us draw the phase portrait of $F_0(y, x) = -I$:

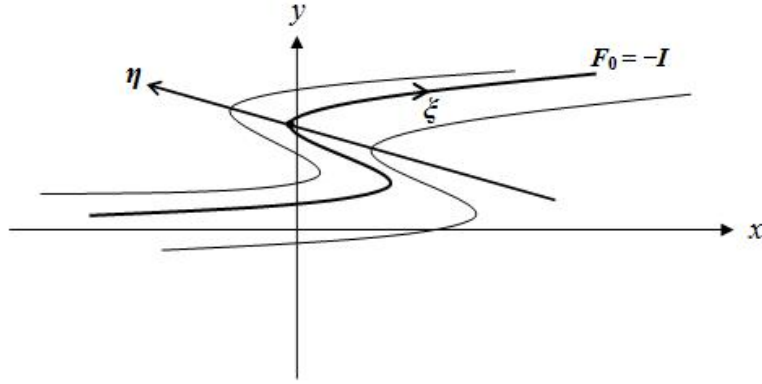


Figure 13

and introduce new coordinates (ξ, η) . Here η is exactly the value of F_0 on level lines $F_0 = \eta$, and ξ is the slow time of motion. The initial section is $\hat{X}(0, I)$, $\hat{Y}(0, I)$ parametrized by I . Then construct a transformation $(\xi, \eta) \mapsto (x, y)$.

Lemma 5.3. In the domain $D_\xi \times D_\eta$, where D_ξ is what mentioned above and D_η is a neighbourhood of the real point $-I_*$, the mapping $(\xi, \eta) \mapsto (x, y)$ is analytic and canonical.

Proof. With map $(\xi, \eta) \mapsto (x, y)$, consider x and y as functions of ξ, η :

$$x = x(\xi, \eta), \quad y = y(\xi, \eta).$$

From the motion above, we have obtained the relations:

$$\frac{\partial x}{\partial \xi} = \frac{\partial F_0}{\partial y}, \quad \frac{\partial y}{\partial \xi} = -\frac{\partial F_0}{\partial x}.$$

Let us differentiate both sides of the Hamiltonian $F_0(y, x, h_0) = \eta$ with respect to η :

$$\frac{\partial F_0}{\partial y} \frac{\partial y}{\partial \eta} + \frac{\partial F_0}{\partial x} \frac{\partial x}{\partial \eta} = 1.$$

Replacing with corresponding derivatives, we get

$$\frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi} \frac{\partial x}{\partial \eta} = 1.$$

That is indeed the Jacobi determinant:

$$\frac{D(x, y)}{D(\xi, \eta)} = \det \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{vmatrix} = 1.$$

Therefore, $(x, y) \mapsto (\xi, \eta)$ is canonical transformation.

□

After this transformation, the new Hamiltonian has the form

$$-I(\eta, \xi, \varphi) = \eta + \varepsilon G_1(\eta, \xi, \varphi).$$

Function G_1 is analytic and has the following estimate:

$$|G_1(\eta, \xi, \varphi)| < \frac{\text{const}}{1 + |\xi|^{2+\nu}}.$$

Also from Cauchy estimate [5], for $\eta \in \tilde{D}_\eta = D_\eta - \delta_\eta$, $\xi \in \tilde{D}_\xi = D_\xi - \delta_\xi$, $\varphi \in \tilde{D}_\varphi$, where δ_η and δ_ξ are positive constants, we obtain

$$\begin{aligned} \left| \frac{\partial G_1(\eta, \xi, \varphi)}{\partial \varphi} \right| &< \frac{\text{const}}{1 + |\xi|^{2+\nu}}, \\ \left| \frac{\partial G_1(\eta, \xi, \varphi)}{\partial \eta} \right| &< \frac{\text{const}}{1 + |\xi|^{2+\nu}}, \\ \left| \frac{\partial G_1(\eta, \xi, \varphi)}{\partial \xi} \right| &< \frac{\text{const}}{1 + |\xi|^{2+\nu}}. \end{aligned}$$

Because of the above estimate, we can take another similar isoenergetic reduction through expressing η , via I , φ , ξ and constant h_0 :

$$-\eta(I, \varphi, \xi) = I + \varepsilon K_1(I, \varphi, \xi).$$

Here $(-\eta)$ is the new Hamiltonian, and ξ is the new slow time. I and φ are conjugate variables.

From implicit function theorem, we can obtain the similar conclusions that function K_1 is analytic in $\widehat{D}_I \times D_\varphi \times D_\xi$ and estimates of K_1 and its derivatives have the following forms:

$$\begin{aligned} |K_1(I, \varphi, \xi)| &< \frac{\text{const}}{1 + |\xi|^{2+\nu}}, \\ \left| \frac{\partial K_1(I, \varphi, \xi)}{\partial I} \right| &< \frac{\text{const}}{1 + |\xi|^{2+\nu}}, \\ \left| \frac{\partial K_1(I, \varphi, \xi)}{\partial \varphi} \right| &< \frac{\text{const}}{1 + |\xi|^{2+\nu}}, \\ \left| \frac{\partial K_1(I, \varphi, \xi)}{\partial \xi} \right| &< \frac{\text{const}}{1 + |\xi|^{2+\nu}}, \end{aligned}$$

in $(\widehat{D}_I - \delta_I) \times \widetilde{D}_\varphi \times \widetilde{D}_\xi$. Here \widehat{D}_I is the same as the domain $-D_\eta$.

In this latest Hamiltonian, $K_0 \equiv I$ and the frequency is $\frac{\partial K_0}{\partial I} = 1$. The level lines

$$\text{Im} \int_0^\xi d\xi_1 = \text{Im} \xi = B = \text{const}, \quad 0 \leq |B| \leq \gamma$$

also lie in the domain D_ξ and have a positive distance from the boundary of D_ξ . The constant γ is the same as introduced in Assumption 3°.

Denoting the new time as $\vartheta = \varepsilon^{-1}\xi$, we can write differential equations of motion:

$$\frac{dI}{d\vartheta} = -\varepsilon \frac{\partial K_1}{\partial \varphi}, \quad \frac{d\varphi}{d\vartheta} = 1 + \frac{\partial K_1}{\partial I}.$$

Now consider the exact solution $I(\vartheta)$, $\varphi(\vartheta)$ with real initial conditions $I(0)$, $\varphi(0)$ at $\vartheta_0 = \varepsilon^{-1}\xi_0 = 0$. We obtain the relation which the solution satisfies as follows:

$$I(\vartheta) = I(0) - \varepsilon \int_0^\vartheta \frac{\partial K_1(I(\vartheta_1), \varphi(\vartheta_1), \varepsilon\vartheta_1)}{\partial \varphi} d\vartheta_1.$$

From the Cauchy criterion [6], we can easily prove the existence of limiting values of $I(\vartheta)$ as $\vartheta \rightarrow \pm\infty$. Moreover, we also know differential equations of motion:

$$\begin{aligned} \frac{d\varphi}{d\vartheta} &= 1 + \varepsilon \frac{\partial K_1}{\partial \varphi} \\ \text{and} \quad \frac{d\varphi}{dt} &= \frac{\partial H_0}{\partial I} + \varepsilon \frac{\partial H_1}{\partial I}. \end{aligned}$$

It is evident that $\varphi \rightarrow \pm\infty$ as $\vartheta \rightarrow \pm\infty$. Meanwhile, $t \rightarrow \pm\infty$. Therefore, the limits when $t \rightarrow \pm\infty$ are equal to those when $\vartheta \rightarrow \pm\infty$. Let us define values

$$I_\pm = \lim_{t \rightarrow \pm\infty} I(t), \quad \Delta I = I_+ - I_-.$$

Now the system under consideration is described by the Hamiltonian

$$-\eta(I, \varphi, \xi) = I + \varepsilon K_1(I, \varphi, \xi),$$

where I and φ are conjugate canonical variables. The action-angle variables I, φ of the unperturbed problem are defined in region $\widehat{D}_I \times D_\varphi$. ξ is the slow time variable defined in region D_ξ and time $\vartheta = \varepsilon^{-1}\xi$. The function K_1 is an analytic, 2π -periodic in φ function. The frequency of unperturbed motion is 1. We have proved the following properties:

1°. The functions $K_0 = I, K_1$ can be analytically extended into a complex domain $\widehat{D}_I \times D_\varphi \times D_\xi$, where \widehat{D}_I is some neighbourhood of a given real point I_* , D_φ is a strip of some fixed width about the real axis, and D_ξ is a strip

$$D_\xi = \left\{ \xi : \begin{cases} |\operatorname{Im} \xi| < \sigma_1 + \delta_1 |\operatorname{Re} \xi|, & |\operatorname{Re} \xi| > c_5 \Gamma \\ |\operatorname{Im} \xi| < \sigma_2, & |\operatorname{Re} \xi| \leq c_5 \Gamma \end{cases} \right\}$$

where $\sigma_1 + \delta_1 \cdot c_5 \Gamma = \sigma_2$. The function K_1 satisfies the estimate

$$|K_1(I, \varphi, \xi)| < \frac{C}{1 + |\xi|^{2+\nu}}.$$

Here σ, δ, C, ν are positive constant.

2°. The level lines

$$\operatorname{Im} \int_0^\xi d\xi_1 = \operatorname{Im} \xi = B = \text{const}, \quad 0 \leq |B| \leq \gamma$$

lie in the domain D_ξ and have a positive distance from the boundary of D_ξ .

Consider a solution $I(\vartheta), \varphi(\vartheta)$ of the Hamiltonian system with real initial conditions $I(0), \varphi(0)$ at $\vartheta = \vartheta_0 = \varepsilon^{-1}\xi_0 = 0$. For

$$I_\pm = \lim_{\vartheta \rightarrow \pm\infty} I(\vartheta), \quad \Delta I = I_+ - I_-,$$

referring to the paper [2], we have the estimate of ΔI that is exponentially small:

$$\Delta I = O(e^{-\frac{\gamma}{\varepsilon}}).$$

Actually, all variables I, φ, y and x mentioned before should contain the bar symbol “ $\bar{}$ ”. We obtain in fact the conclusion that

$$\Delta \bar{I} = \bar{I}_+ - \bar{I}_- = O(e^{-\frac{\gamma}{\varepsilon}}), \quad \gamma = \text{const} > 0.$$

Now return the bar. Recall that from the original Hamiltonian $E(p, q, y, x)$, we transform to two forms of Hamiltonian. With generating function $S(I, q, y, x)$ where y, x are considered as parameters, we obtain $H_0(I, y, x)$ and with $\varepsilon^{-1}\bar{y}x + S(\bar{I}, q, \bar{y}, x)$ we obtain $H_0(\bar{I}, \bar{y}, \bar{x}) + \varepsilon H_1(\bar{I}, \bar{\varphi}, \bar{y}, \bar{x})$. Since the limiting values of \bar{I} exist, now let us prove the lemma that the limiting value of I is equal to that of \bar{I} and thus exists.

Lemma 5.4.

$$\lim_{t \rightarrow \pm\infty} I(t) = \lim_{t \rightarrow \pm\infty} \bar{I}(t).$$

Proof. With generating function $S(p, q, y, x)$, the Hamiltonian $E(p, q, y, x)$ is canonically transformed to $H_0(I, y, x)$ and $p = \frac{\partial S(I, q, y, x)}{\partial q}$.

As we know

$$\frac{\partial S}{\partial x} = \int_0^t \left(\left\langle \frac{\partial E}{\partial x} \right\rangle - \frac{\partial E}{\partial x} \right) dt_1,$$

so

$$\frac{\partial S}{\partial x} \rightarrow 0, \quad \text{since } \frac{\partial E}{\partial x} \rightarrow 0, \quad \text{as } x \rightarrow \pm\infty.$$

On the other hand, with generating function $\varepsilon^{-1}\bar{y}x + S(\bar{I}, q, \bar{y}, x)$, the Hamiltonian $E(p, q, y, x)$ is transformed to $H_0(\bar{I}, \bar{y}, \bar{x}) + \varepsilon H_1(\bar{I}, \bar{\varphi}, \bar{y}, \bar{x})$ and $p = \frac{\partial S(\bar{I}, q, \bar{y}, x)}{\partial q}$. Also we know

$$H_1 = - \frac{\partial H_0}{\partial \bar{x}} \bigg|_{\tilde{x}} \frac{\partial S}{\partial \bar{y}} + \frac{\partial E}{\partial y} \bigg|_{\tilde{y}} \frac{\partial S}{\partial x},$$

where \tilde{x} and \tilde{y} are some intermediate values. As $\frac{\partial H_0}{\partial \bar{x}} \rightarrow 0$, $\frac{\partial S}{\partial x} \rightarrow 0$, thus $H_1 \rightarrow 0$, as $x \rightarrow \pm\infty$.

From assumption $\left(\frac{\partial H_0}{\partial \bar{y}}\right)^2 + \left(\frac{\partial H_0}{\partial \bar{x}}\right)^2 > \text{const}$, and $\frac{\partial H_0}{\partial \bar{x}} \rightarrow 0$, as $\bar{x} \rightarrow \pm\infty$, we know that $\left|\frac{\partial H_0}{\partial \bar{y}}\right| > \text{const}$ for big $|\bar{x}|$. Thus, when $|\bar{x}| > c_6$, $\left|\frac{\partial H_0}{\partial \bar{y}}\right| > c_7^{-1}$, where c_6, c_7 are positive constants. From $\dot{\bar{y}} = \frac{d\bar{y}}{dt} = -\varepsilon \left(\frac{\partial H_0}{\partial \bar{x}} + \varepsilon \frac{\partial H_1}{\partial \bar{x}} \right)$, $\dot{\bar{x}} = \frac{d\bar{x}}{dt} = \varepsilon \left(\frac{\partial H_0}{\partial \bar{y}} + \varepsilon \frac{\partial H_1}{\partial \bar{y}} \right)$, the phase portrait of $H_0 + \varepsilon H_1 = \text{const}$ in (\bar{x}, \bar{y}) plane will close to that of the adiabatic approximation $H_0 = \text{const}$ for $|\bar{x}| \leq c_6 + 1$.



Figure 14

Thus for some time moments t_+ and t_- , we have

$$\bar{x}(t_+) > c_6, \quad \bar{x}(t_-) < -c_6.$$

Then for $t > t_+$, we have

$$\dot{x} = \varepsilon \left(\frac{\partial H_0}{\partial \bar{y}} + \varepsilon \frac{\partial H_1}{\partial \bar{y}} \right) > \varepsilon (c_7^{-1} + O(\varepsilon)) > \frac{1}{2} \varepsilon c_7^{-1}.$$

Therefore, $\bar{x}(t) \rightarrow +\infty$ as $t \rightarrow +\infty$. Similarly, $\bar{x}(t) \rightarrow -\infty$ as $t \rightarrow -\infty$.

From

$$y = \bar{y} + \varepsilon \frac{\partial S(\bar{I}, q, \bar{y}, x)}{\partial x} = \bar{y} + \varepsilon V(\bar{I}, \bar{\varphi}, \bar{y}, x),$$

taking limiting values on both sides as $t \rightarrow \pm\infty$, we obtain $y - \bar{y} \rightarrow 0$.

We also know that

$$\begin{aligned} p &= P(I, \varphi, y, x) = P(\bar{I}, \bar{\varphi}, \bar{y}, x), \\ q &= Q(I, \varphi, y, x) = Q(\bar{I}, \bar{\varphi}, \bar{y}, x). \end{aligned}$$

As $y - \bar{y} \rightarrow 0$, it is evident that $I - \bar{I} \rightarrow 0$, $\varphi - \bar{\varphi} \rightarrow 0$ as $t \rightarrow \pm\infty$.

Therefore,

$$\lim_{t \rightarrow \pm\infty} I(t) = \lim_{t \rightarrow \pm\infty} \bar{I}(t).$$

□

Thus I and \bar{I} have the same limit, and $\Delta I = \Delta \bar{I}$. Therefore, we have the following corollary:

Corollary 5.2. *The estimate*

$$\Delta I = O(e^{-\frac{\gamma}{\varepsilon}})$$

is valid.

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